HOMEWORK 9 – ANSWERS TO (MOST) PROBLEMS

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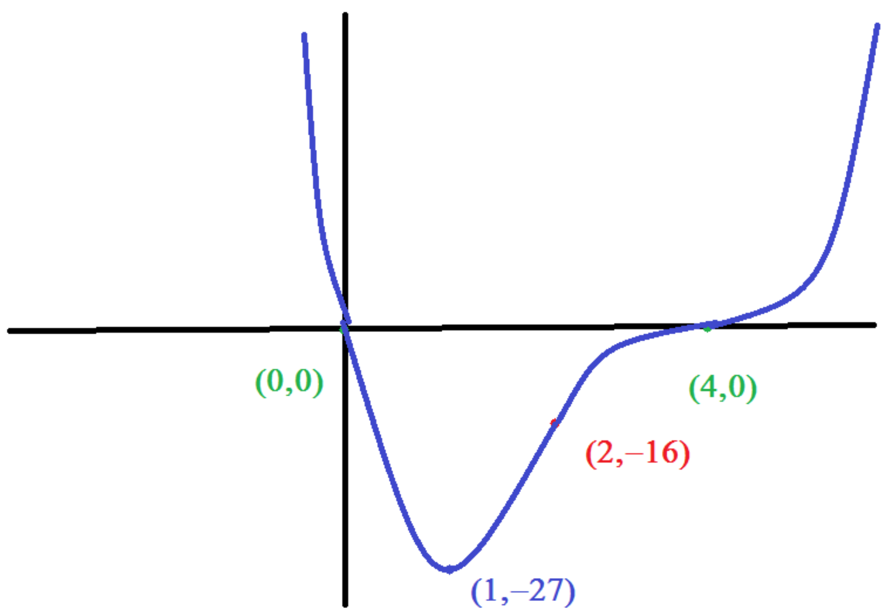
Section 4.5: Summary of curve sketching

4.5.5.

D : $\mathbb{R}$
I : $x-$intercepts: $0, 4$, $y-$intercept: 0
S : None
A : None, but $\lim_{x\to\pm\infty} f(x) = \infty$
I : $f'(x) = (x-4)^3 + 3x(x-4)^2 = (x-4)^2(x-4+3x) = (x-4)^2(4x-4) = 4(x-4)^2(x-1)$; f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$; Local minimum $f(1) = -27$
C : $f''(x) = 8(x-4)(x-1) + 4(x-4)^2 = 4(x-4)(2x-2+x-4) = 4(x-4)(3x-6) = 12(x-4)(x-2)$; f is concave up on $(-\infty, 2)$, concave down on $(2, 4)$, and concave up on $(4, \infty)$. Inflection points: $(2, -16)$, $(4, 0)$

1A/Math 1A - Fall 2013/Homeworks/Quartic.png

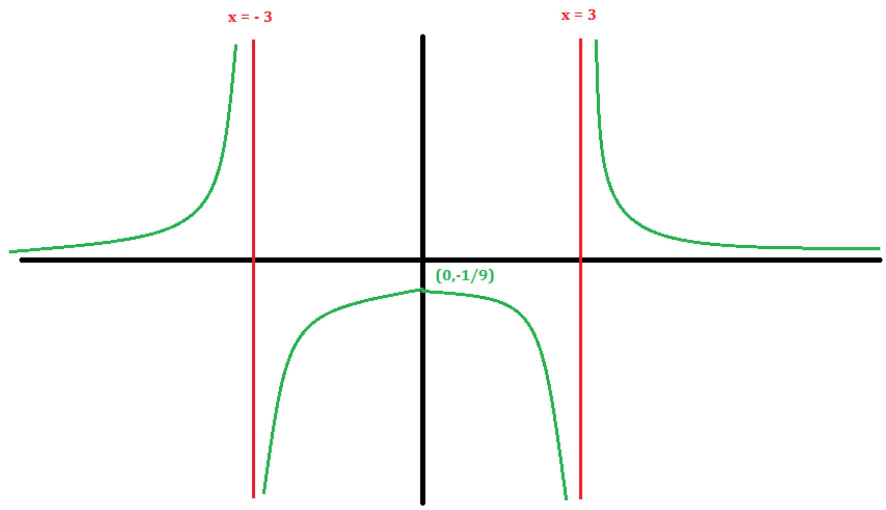


Date: Friday, November 8th, 2013.

4.5.13.

D : $\mathbb{R} - \{\pm 3\}$
I : No $x-$intercepts, $y-$intercept: $y = -\frac{1}{9}$
S : f is even
A : Horizontal Asymptote $y = 0$ (at $\pm\infty$), Vertical Asymptotes $x = \pm 3$
I : $f'(x) = -\frac{2x}{(x^2-9)^2}$; f is increasing on $(-\infty, -3) \cup (-3, 0)$ and decreasing on $(0, 3) \cup (3\infty)$. Local maximum of $\frac{-1}{9}$ at 0.
C : $f''(x) = 6\frac{x^2+3}{(x^2-9)^3}$; f is concave up on $(-\infty, -3) \cup (3, \infty)$ and concave down on $(-3, 3)$; No inflection points

1A/Archive/Homeworks - Spring 2011/hw10graph1.png



4.5.45.

D : $x > 0$
I : No $x-$intercept because $f(x) > 0$ for all x (see Increasing/Decreasing section). No $y-$intercept (not defined at 0)
S : No symmetries
A : Vertical asymptote $x = 0$, No Horizontal Asymptote, because:

$$\lim_{x\to\infty} x - \ln(x) = \lim_{x\to\infty} x\left(1 - \frac{\ln(x)}{x}\right) = \infty(1-0) = \infty$$

Also no slant asymptote, because if there were such a slant asymptote $y = ax + b$, then:

$$a = \lim_{x\to\infty} \frac{\ln(x) - x}{x} = -1$$

And then:

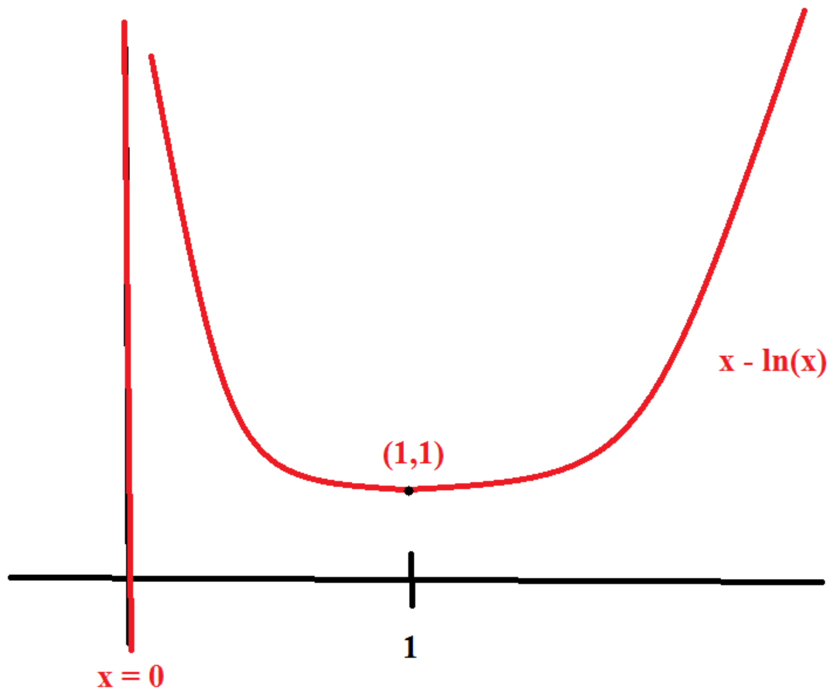
$$b = \lim_{x\to\infty} (\ln(x) - x) - (-1)x = \lim_{x\to\infty} \ln(x) = \infty$$

which is a contradiction!

I : $f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$, so decreasing on $(0,1)$ and increasing on $(1,\infty)$; local minimum $f(x) = 1$. In particular $f(x) \geq 1$ for all x, and so $f(x) > 0$ (hence no $x-$intercept)

C : $f''(x) = \frac{1}{x^2}$, concave up on $(0,\infty)$; No inflection points.

1A/Math 1A - Fall 2013/Homeworks/x - ln(x).png



4.5.71. <u>At ∞:</u>

Suppose the slant asymptote is $y = ax + b$, then:

$$a = \lim_{x\to\infty} \frac{x - \tan^{-1}(x)}{x} = \lim_{x\to\infty} 1 - \frac{\tan^{-1}(x)}{x} = 1 - \frac{\frac{\pi}{2}}{\infty} = 1$$

$$b = \lim_{x\to\infty} x - \tan^{-1}(x) - x = \lim_{x\to\infty} -\tan^{-1}(x) = -\frac{\pi}{2}$$

Hence $x - \tan^{-1}(x)$ has a slant asymptote of $y = x - \frac{\pi}{2}$ at ∞

<u>At $-\infty$:</u>

Suppose the slant asymptote is $y = ax + b$, then:

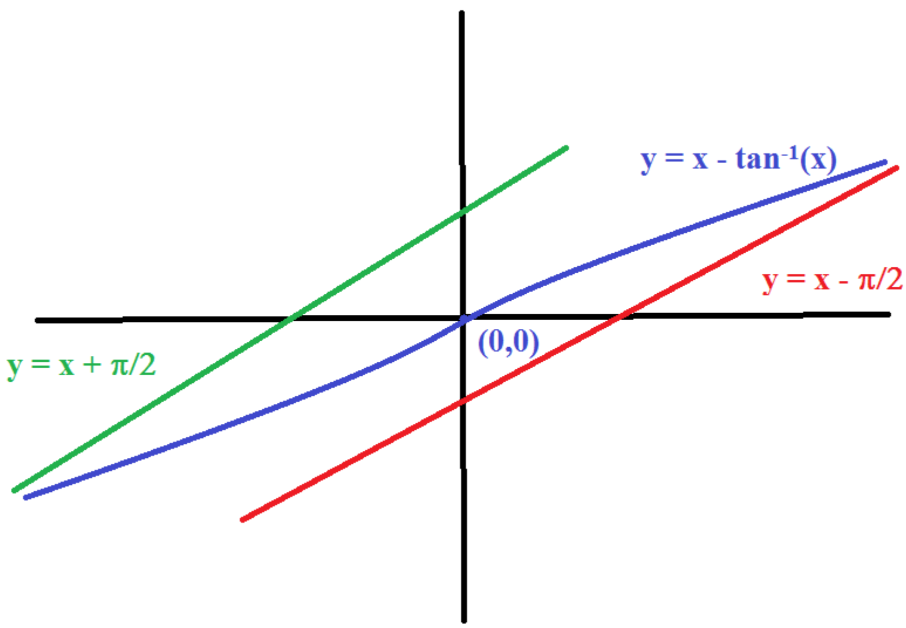
$$a = \lim_{x\to-\infty} \frac{x - \tan^{-1}(x)}{x} = \lim_{x\to-\infty} 1 - \frac{\tan^{-1}(x)}{x} = 1 - \frac{\frac{\pi}{2}}{-\infty} = 1$$

$$b = \lim_{x \to -\infty} x - \tan^{-1}(x) - x = \lim_{x \to -\infty} -\tan^{-1}(x) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Hence $x - \tan^{-1}(x)$ has a slant asymptote of $y = x + \frac{\pi}{2}$ at $-\infty$

- D $Dom = \mathbb{R}$
- I $y-$intercept: $f(0) = 0$, $x-$intercept: 0 (there are no others, because f is increasing; see Increasing/Decreasing section)
- S No symmetries
- A No vertical asymptotes (f is defined everywhere), Slant Asymptotes $y = x - \frac{\pi}{2}$ at ∞, $y = x + \frac{\pi}{2}$ at $-\infty$; No H.A. because there are already two S.A.
- I $f'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} \geq 0$, so f is increasing everywhere; No local max/min
- C $f''(x) = \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$, so f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Inflection point $(0, f(0)) = (0, 0)$

1A/Math 1A - Fall 2013/Homeworks/x - arctan(x).png



Section 4.7: Optimization Problems

4.7.3.

- Want to minimize $x + y$
- But $xy = 100$, so $y = \frac{100}{x}$, so $x + y = x + \frac{100}{x}$
- Let $f(x) = x + \frac{100}{x}$
- $x > 0$ (x is positive)
- $f'(x) = 0 \Leftrightarrow 1 - \frac{100}{x^2} = 0 \Leftrightarrow x^2 = 100 \Leftrightarrow x = 10$
- By FDTAEV, $x = 10$ is the absolute minimizer of f
- Answer: $\boxed{x = 10,\ y = \frac{100}{10} = 10}$

4.7.14.

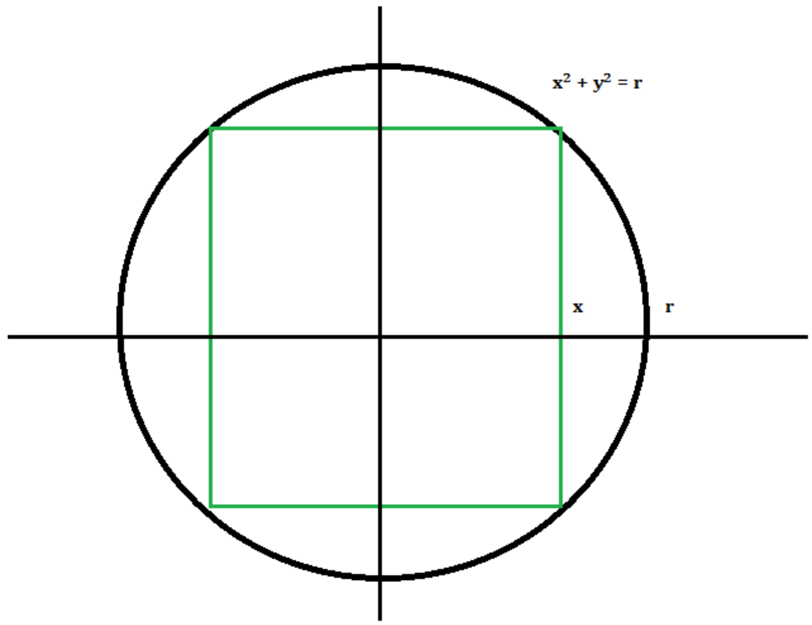
- Want to minimize $S = x^2 + 4xh$ (where x is the length of the base-side and h is the height)
- However, $V = x^2 h = 32000$, so $h = \frac{32000}{x^2}$, so $x^2 + 4xh = x^2 + 4x\frac{32000}{x^2} = x^2 + \frac{128000}{x}$
- Let $f(x) = x^2 + \frac{128000}{x}$
- $x > 0$
- $f'(x) = 0 \Leftrightarrow 2x - \frac{128000}{x^2} = 0 \Leftrightarrow 2x^3 = 128000 \Leftrightarrow x = \sqrt[3]{64000} = 40$
- By FDTAEV, $x = 40$ is the absolute minimizer of f
- Answer: $\boxed{x = 40,\ h = \frac{32000}{(40)^2} = \frac{32000}{1600} = 20}$

4.7.21.

- We have $D = \sqrt{(x-1)^2 + y^2}$, so $D^2 = (x-1)^2 + y^2$
- But $y^2 = 4 - 4x^2$, so $D^2 = (x-1)^2 + 4 - 4x^2$
- Let $f(x) = (x-1)^2 + 4 - 4x^2$
- No constraints
- $f'(x) = 2(x-1) - 8x = -6x - 2 = 0 \Leftrightarrow x = -\frac{1}{3}$
- By the FDTAEV, $x = -\frac{1}{3}$ is the maximizer of f.
- Since $y^2 = 4 - 4x^2$, we get $y^2 = 4 - \frac{4}{9} = \frac{32}{9}$, so $y = \pm\sqrt{\frac{32}{9}} = \pm\frac{4\sqrt{2}}{3}$
- Answer: $\boxed{\left(-\frac{1}{3}, -\frac{4\sqrt{2}}{3}\right) \text{ and } \left(-\frac{1}{3}, \frac{4\sqrt{2}}{3}\right)}$

4.7.23. Picture:

1A/Math 1A - Fall 2013/Solution Bank/hw10opt1.png



- We have $A = xy$, but the trick here again is to maximize $A^2 = x^2y^2$ (thanks for Huiling Pan, a former student of mine, for this suggestion!)
- But $x^2 + y^2 = r^2$, so $y^2 = r^2 - x^2$, so $A^2 = x^2(r^2 - x^2) = x^2r^2 - x^4$
- Let $f(x) = x^2r^2 - x^4$
- Constraint $0 \leq x \leq r$ (look at the picture)
- $f'(x) = 2xr^2 - 4x^3 = 0 \Leftrightarrow x = 0$ or $x = \frac{r}{\sqrt{2}}$
- By the closed interval method, $x = \frac{r}{\sqrt{2}}$ is a maximizer of f (basically $f(0) = f(r) = 0$
- Answer: $\boxed{x = \frac{r}{\sqrt{2}}, \ y = \sqrt{r^2 - \frac{r^2}{2}} = \frac{r}{\sqrt{2}}}$

4.7.32.

- $A = 2rh + \frac{1}{2}\pi r^2$
- But $P = \pi r + 2r + 2h = 30$, so $h = 15 - r - \frac{\pi}{2}r$
- Let $f(r) = 2r\left(15 - r - \frac{\pi}{2}r\right) + \frac{1}{2}\pi r^2 = 30r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2 = 30r - 2r^2 - \frac{\pi}{2}r^2$
- Constraint $r > 0$
- $f'(r) = 30 - 4r - \pi r = 0 \Leftrightarrow r = \frac{30}{\pi+4}$
- By FDTAEV, $r = \frac{30}{\pi+4}$ is the minimizer of f
- $\boxed{r = \frac{30}{\pi+4}, \ h = 15 - \frac{30}{\pi+4} - \frac{15\pi}{\pi+4} = \frac{30}{\pi+4} = r}$

4.7.48.

- Let t_{AB} be the time spent rowing from A to B and t_{BC} be the time spent walking from B to C
- By the formula time $= \frac{\text{distance}}{\text{velocity}}$, we have:

$$t_{AB} = \frac{AB}{2} = \frac{\cos(\theta)AC}{2} = \frac{4\cos(\theta)}{2} = 2\cos(\theta)$$

$$t_{BC} = \frac{BC}{4} = \frac{2 \times \angle BOC}{4} = \frac{2 \times 2\theta}{4} = \theta$$

(here O is the origin; it is a geometric fact that $\angle BOC = 2\angle BAC$)

- Let $f(\theta) = 2\cos(\theta) + \theta$
- Constraint: $0 \leq \theta \leq \frac{\pi}{2}$ (see the picture!)
- $f'(\theta) = -2\sin(\theta) + 1 = 0 \Leftrightarrow \sin(\theta) = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$
- $f(0) = 2$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ and $f\left(\frac{\pi}{3}\right) = 2 \times \frac{\sqrt{3}}{2} + \frac{\pi}{3} = \sqrt{3} + \frac{\pi}{3}$.

By the **closed interval method**, $\theta = \frac{\pi}{2}$ is an absolute minimizer.

- Therefore, she should just walk! (which makes sense because she walks much faster than she rows!)

4.7.61.

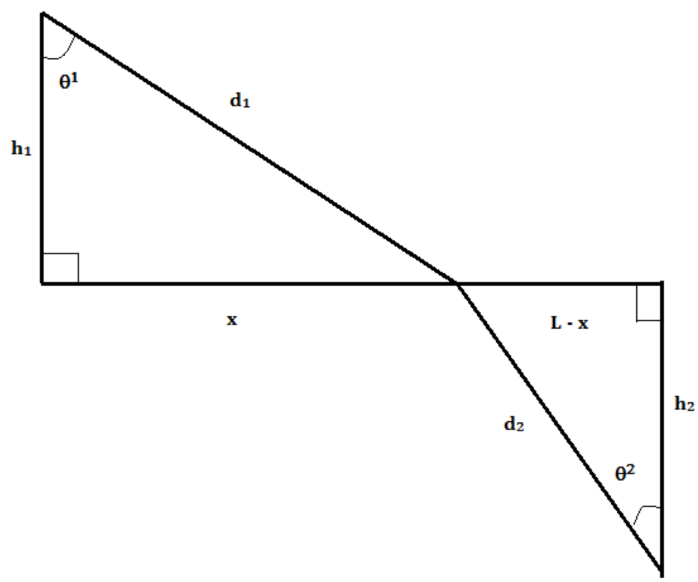
(a) $p(x) = 550 - \frac{x}{10}$ (Basically imitate Example 6 on page 331)

(b) The revenue function is $R(x) = xp(x) = 550x - \frac{x^2}{10}$. $R'(x) = 0 \Leftrightarrow 550 = \frac{x}{5} \Leftrightarrow x = 2750$, and the corresponding price is $p(2750) = 550 - 275 = 275$ and the rebate is $450 - 275 = 175$ dollars

(c) Here the profit function is $P(x) = R(x) - C(x) = 550x - \frac{x^2}{10} - 68000 - 150x$. $P'(x) = 0 \Leftrightarrow 550 - \frac{x}{5} - 150 = 0 \Leftrightarrow x = 2000$, so the corresponding price is $p(2000) = 550 - 200 = 350$, so the corresponding rebate is $450 - 350 = 100$ dollars

4.7.67. The picture is as follows:

1A/Math 1A - Fall 2013/Solution Bank/hw10opt2.png



Here, h_1 and h_2 and L are fixed, but x varies.
Now the total time taken is $t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2}$.

Now, by the Pythagorean theorem: $d_1 = \sqrt{x^2 + h_1^2}$ and $d_2 = \sqrt{(L-x)^2 + h_2^2}$, so we get:

$$t(x) = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(L-x)^2 + h_2^2}}{v_2}$$

And

$$t'(x) = \frac{x}{v_1\sqrt{x^2 + h_1^2}} + \frac{x - L}{v_2\sqrt{(L-x)^2 + h_2^2}} = \frac{x}{v_1 d_1} + \frac{x - L}{v_2 d_2}$$

Setting $t'(x) = 0$ and cross-multiplying, we get:

$$v_1 d_1 (L - x) = v_2 d_2 x$$

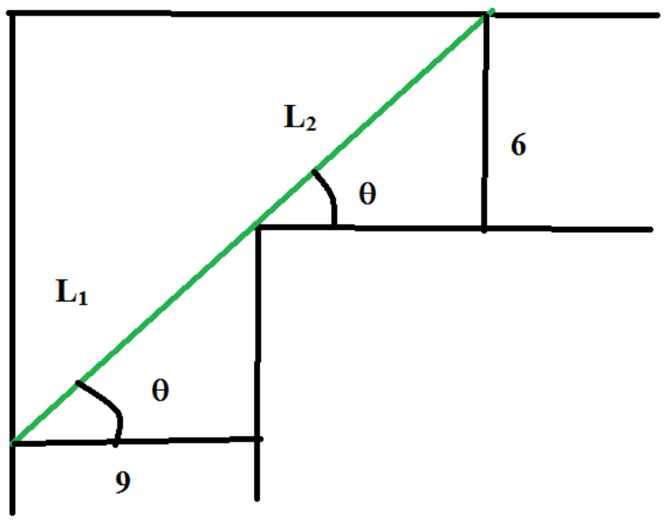
So, by definition of $\sin(\theta_1)$ and $\sin(\theta_2)$), we get:

$$\frac{v_1}{v_2} = \frac{d_2 x}{(L-x)d_1} = \frac{\frac{x}{d_1}}{\frac{L-x}{d_2}} = \frac{\sin(\theta_1)}{\sin(\theta_2)}$$

Note: Thank you Brianna Grado-White (a former student of mine) for a solution to this problem!

4.7.70. The picture is as follows (Note that the two $\theta-$s are indeed the same!)

1A/Math 1A - Fall 2013/Homeworks/Pipe.png



- We want to minimize $L_1 + L_2$
- $\cos(\theta) = \frac{L_1}{9}$, so $L_1 = \frac{9}{\cos(\theta)}$, $\sin(\theta) = \frac{L_2}{6}$, so $L_2 = \frac{6}{\sin(\theta)}$
- Let $f(\theta) = \frac{9}{\cos(\theta)} + \frac{6}{\sin(\theta)}$
- Constraint: $0 < \theta < \frac{\pi}{2}$ (Notice that at 0 and $\frac{\pi}{2}$, we can't carry the pipe horizontally around the corner; it would break at that corner)
- $f'(\theta) = \frac{9\sin(\theta)}{\cos^2(\theta)} + \frac{-6\cos(\theta)}{\sin^2(\theta)} = \frac{9\sin^3(\theta) - 6\cos^3(\theta)}{\cos^2(\theta)\sin^2(\theta)} = 0$

  $\Leftrightarrow 9\sin^3(\theta) - 6\cos^3(\theta) = 0 \Leftrightarrow \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^3 = \frac{6}{9} = \frac{2}{3} \Leftrightarrow \tan^3(\theta) = \frac{2}{3} \Leftrightarrow \theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$
- By FDTAEV, $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ is the absolute minimizer of f
- Answer: $\frac{9}{\cos(\theta)} + \frac{6}{\sin(\theta)}$, where $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ (if you want to, you can simplify this using the triangle method: $\frac{1}{\cos(\tan^{-1}(x))} = \sqrt{1+x^2}$ and $\frac{1}{\sin(\tan^{-1}(x))} = \frac{\sqrt{1+x^2}}{x}$, but I think this is enough torture for now :)

Disclaimer: This last problem problem is kinda ridiculous, and it took me an hour to figure this out! However, that doesn't mean that you shouldn't do it!

Section 4.9: Antiderivatives

4.9.16. $R(\theta) = \sec(\theta) - 2e^{\theta}$

4.9.35. $f(x) = -2\sin(t) + \tan(t) + C$, but $4 = f(\frac{\pi}{3}) = -\sqrt{3} + \sqrt{3} + C = C$, so $f(x) = -2\sin(t) + \tan(t) + 4$

4.9.63. $a(t) = 10\sin(t) + 3\cos(t)$, so $v(t) = -10\cos(t) + 3\sin(t) + A$, so $s(t) = -10\sin(t) - 3\cos(t) + At + B$

Now, $s(0) = 0$, but $s(0) = -10(0) - 3(1) + A(0) + B$, so $-3 + B = 0$, so $B = 3$

So $s(t) = -10\sin(t) - 3\cos(t) + At + 3$

Moreover, $s(2\pi) = 12$, but $s(2\pi) = -10(0) - 3(1) + A(2\pi) + 3 = A(2\pi)$, so $A(2\pi) = 12$, so $A = \frac{12}{2\pi} = \frac{6}{\pi}$

So altogether, you get: $\boxed{s(t) = -10\sin(t) - 3\cos(t) + \frac{6}{\pi}t + 3}$

4.9.76. Suppose the acceleration of the car is $a(t) = A$. Then $v(t) = At + B$ and $s(t) = \frac{A}{2}t^2 + Bt + C$.

However, at $t = 0$, the car is moving at 100 km/h, so $v(0) = 100$, so $B = 100$, hence $v(t) = At + 100$ and $s(t) = \frac{A}{2}t^2 + 100t + C$.

Moreover, at $t = 0$, the car is at its initial position 0, so $s(0) = 0$, so $C = 0$, hence $s(t) = \frac{A}{2}t^2 + 100t$

Now let t^* be the time needed to real the pile-up.

We want the car to have 0 velocity at t^*, hence $v(t^*) = 0$, hence $At^* + 100 = 0$, so $At^* = -100$

Moreover, we want $s(t^*) = 80\text{m} = 0.08$ km, so $\frac{A}{2}(t^*)^2 + 100t^* = 0.08$, but using the fact that $At^* = -100$, this just becomes: $\frac{-100t^*}{2} + 100t^* = 0.08$, so $50t^* = 0.08$, so $t^* = \frac{1}{625}$.

Therefore $A = -\frac{100}{t^*} = -100 \times 625 = -62500\ km/h^2$, so the answer is $\boxed{62500\ km/h^2}$.